cell are dependent upon the dimensions of the latter, its plane of division and the size and location of the nucleus. Thus, in very small, isodiametric cells, having a large centrally located nucleus, the cell plate quickly intersects the walls of the cell, without any extensive laterial growth—by means of additional peripheral fibers—of the spindle. In somewhat larger elements there is sufficient room for the process of cytokinesis to reach the "halo" stage before the cell plate intersects the sides of the cell. Only in elongated or much flattened elements, i.e., cells which are not isodiametric, is it possible for the phenomenon of cell plate formation to pass through the "spindle," "disk," "halo" and "frame" stages and finally form two entirely separate aggregations of kinoplasmic fibrillae, the kinoplasmasomes.

It is evident, accordingly, that the remarkable type of cell plate formation which was described in a former note, is not an isolated or unusual phenomenon, but is of frequent occurrence in the somatic tissue of the higher plants, gymnosperms and angiosperms. It promises to be significant in any general discussion concerning the dynamics of cytokinesis and karyokinesis.

[1] Phenomena of cell division in the cambium of arborescent gymnosperms and their cytological significance, these PROCEEDINGS, **5,** 1919 (283–285).

[2] Bailey, I. W., The significance of the cambium in the study of certain physiological problems, *J. Gen. Physiol.* Ined.

[3] The average length of the cambial initials in 152 stems of gymnosperms was computed as approximately 3400 micra; in 275 stems of dicotyledons as approximately 600 micra.

[4] The word fibers is used without reference to whether the fibrillae are true threads or lines of protoplasmic streaming in the living cell.

[5] Treub, M., "Quelques recherches sur le rôle du noyau dans la division des cellules végétales," *Verh. K. Akad. Wetensch. Amsterdam,* **19,** 1879 (1–35).

[6] Strasburger, E., *Zellbildung und Zelltheilung,* 3 aufl, Jena, 1880.

[7] Schürhoff, P., "Das Verhalten des Kernes im Wundgewebe," *Beih. Bot. Central.,* **19,** 1906 (359–382).

FUNCTIONALS INVARIANT UNDER ONE-PARAMETER CONTINUOUS GROUPS OF TRANSFORMATIONS IN THE SPACE OF CONTINUOUS FUNCTIONS

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1. *Some general notions.*—It is intended in this abstract to give a few examples of one-parameter continuous groups of transformations in function space which are analogues of well-known groups in the space of a finite number of dimensions, and to exhibit in each case a functional invariant in terms of which every invariant of the group (with suitable restrictions) is expressible. Some of these groups have already been con-

sidered by G. Kowalewski,[1] although, so far as the writer knows, no attempt has been made to discover the invariants of the group.

Let u or $u(\xi)$ be a real continuous function defined on the interval $a \leqq \xi \leqq b$, and τ a real variable ranging over the interval $|\tau - \tau_0| \leqq c$. Consider also a functional operation $F[x,u,z]$ which, for every element (x, u, τ) of the set defined by

$$a \leqq x \leqq b, \quad \max_{\xi} |u(\xi) - u_0(\xi)| \leqq \alpha, \quad |\tau - \tau_0| \leqq c$$

determines a real number. If, furthermore, F is a continuous functional[2] of its arguments, then for each τ, one may regard F as a transformation taking the continuous function u into another continuous function $\bar{u}$ of the variable x,

$$\bar{u}(x) = F[x,u,\tau] \tag{1}$$

As usual, the transformations (1) are said to form a group if the product of every two transformations of the set is in the set, and if, furthermore, the set contains the identity transformation and for every transformation the corresponding inverse transformation.

An infinitesimal transformation is defined by the equation

$$\frac{\partial u(x,\tau)}{\partial \tau} = f[x,u], \tag{2}$$

where

$$f[x,u] = \frac{\partial}{\partial \tau} F[x,u,\tau]_{\iota = \iota_0}$$

τ_0 denoting the parameter giving the identity transformation of (1). If one supposes that the functionals have suitable properties of continuity and differentiability,[2] one can show that equation (2) determines a one-parameter continuous group of transformations.

By a method exactly like that employed by Lie it can be shown that the functionals $G(u)$ which are finite invariants (*finite* is here used in contradistinction from *differential*) of the group (1) may be found from the solutions of the Stieltjes integral equation

$$\int_a^b f[x,u] d_x \varphi[x,u] = 0 \tag{3}$$

where the φ is defined by

$$\int_0^1 u(x) d_x \varphi[x,u] = \Phi[u],$$

and Φ is the Frêchet differential[3] of G. It can be shown that if the functional f has the properties already referred to, solutions of (3) exist.[4]

2. *Examples.*—The set of transformations

$$\bar{u}(x) = u(x) + \alpha(x)\tau = F[x,u,\tau]$$

where $\alpha(x)$ is an arbitrary but specified continuous function evidently

represents a group in function space with $\tau = 0$ as the identity parameter. The infinitesimal transformation is given by the equation

$$\frac{\partial u(x,\tau)}{\partial \tau} = \alpha(x).$$

Furthermore, every invariant satisfying suitable restrictions of continuity and differentiability may be expressed as a functional of the invariant

$$I[x,u] = \begin{vmatrix} u(x) & \int_a^b u(\xi)d\xi \\ \alpha(x) & \int_a^b \alpha(\xi)d\xi \end{vmatrix}.$$

The transformations

$$\bar{u}(x) = \tau u(x) = F[x,u,\tau]$$

form a group. The infinitesimal transformation is

$$\frac{\partial u(x,\tau)}{\partial \tau} = u(x),$$

and every invariant is a functional of[5]

$$I[x,u] = \frac{u(x)}{\int_a^b u(\xi)d\xi}.$$

It is easily verified that the transformations

$$\bar{u}(x) = u(x)e^{-\tau} + (e^{\tau} - e^{-\tau})\int_0^1 u(\xi)d\xi = F[x,u,\tau]$$

form a group with $\tau = 0$ giving the identity transformation. The infinitesimal transformation is defined by the integro-differential equation

$$\frac{\partial u(x,\tau)}{\partial \tau} = -u(x) + 2\int_0^1 u(\xi)d\xi.$$

Every invariant is a functional of

$$I[x,u] = u(x)\int_0^1 u(\xi)d\xi - \left\{\int_0^1 u(\xi)d\xi\right\}^2.$$

Consider the group defined by the infinitesimal transformation

$$\frac{\partial u(x,\tau)}{\partial \tau} = \int_0^{2\pi} \sin \sigma(x - \xi)u(\xi)d\xi$$

where σ is an odd integer. It can be shown[6] that the finite group may be put in the form

$$\bar{u}(\quad = u(x) + \frac{1}{2\pi}\Big[e^{-\sigma ix}(e^{-\pi i\tau} - 1)\int_0^{2\pi} e^{\sigma i\xi}u(\xi)d\xi$$
$$+ e^{\sigma ix}(e^{\pi ix} - 1)\int_0^{2\pi} e^{-\sigma i\xi}u(\xi)d\xi\Big] = F[x,u,\tau]$$

Every invariant is a functional of the invariant

$$I[x,u] = u(x) + \frac{1}{2\pi}\left[1 - \int_0^{2\pi} e^{\sigma\iota\xi} u(\xi)d\xi\right]\left[e^{-\sigma i x} - e^{\sigma i x}\int_0^{2\pi} e^{-\sigma\iota\xi} u(\xi)d\xi\right].$$

Some interesting special invariants are obtained when the particular invariant functionals

$$\int_0^{2\pi} e^{-\sigma i x} I[x,u]dx \text{ and } \int_0^{2\pi}\left\{I[x,u]\right\}^2 dx$$

are taken in which cases one obtains, respectively, the invariants

$$\left[\int_0^{2\pi}\cos\sigma\xi u(\xi)d\xi\right]^2 + \left[\int_0^{2\pi}\sin\sigma\xi\, u(\xi)d\xi\right]^2, \int_0^{2\pi} u^2(x)dx.$$

This group is a special instance of the orthogonal group in function space[1] defined by the infinitesimal transformation

$$\frac{\partial u(x,\tau)}{\partial\tau} = \int_a^b K(x,\xi)u(\xi)d\xi,$$

where $K(x,\xi)$ is a skew-symmetric kernel; i.e., $K(x,\xi) = -K(\xi,x)$. The invariants of this group are quite complicated and will not be given.

As a last example, let a group have the last mentioned infinitesimal transformation where the given kernel function $K(x,\xi)$ is continuous and symmetric. The finite transformations may be put in the form[6]

$$\bar{u}(x) = u(x) + \sum_{n=1}^{\infty}\varphi_n(x)(e^{\frac{\tau}{\lambda_n}} - 1)\int_a^b \varphi_n(\xi)u(\xi)d\xi = F[x,u,\iota\tau],$$

where the constants λ_n are the characteristic numbers of the kernel $K(x,\xi)$ and the functions $\varphi_n(\xi)$, are the corresponding normed orthogonal set of characteristic functions. Every invariant is a functional of

$$I[x,u] = u(x) + \sum_{n=1}^{\infty}\varphi_n(x)\left\{\left[\int_a^b \varphi_n(\xi)u(\xi)d\xi\right]\frac{1 - \left[\int_a^b \varphi_1(\xi)u(\xi)d\xi\right]^{\frac{\lambda_1}{\lambda_n}}}{\left[\int_a^b \varphi_1(\xi)u(\xi)d\xi\right]^{\frac{\lambda_1}{\lambda_n}}}\right\}.$$

If the particular invariant functional

$$\int_a^b I[x,u]\varphi_k(x)dx, \quad k \neq 1,$$

is taken, one obtains the invariants

$$\frac{\int_a^b \varphi_k(\xi)u(\xi)d\xi,}{\left[\int_0^1 \varphi_1(\xi)u(\xi)d\xi\right]^{\frac{\lambda_1}{\lambda_k}}}, k \neq 1.$$

One could now try to extend some other results of the theory of continuous groups to functions both for one parameter and for several parameters. For example, it would be interesting to know what the invariant manifolds of a given group are; also the applications to partial differential equations in infinitely many variables, which the writer is now engaged

in investigating. Finally, groups in which the functions $u(x)$ are discontinuous would seem to be worthy of consideration.

[1] G. Kowalewski, *Sitz. ber. Math. Akad. Wiss. Wien*, **120**, Abteilung IIa^1, IIa^2, 1911 (77–109,1435–72).

[2] See writer's doctor's dissertation (Chicago, 1918).

[3] *Trans. Amer. Math. Soc.*, **15**, 1914*r* (139).

[4] In a paper by the writer soon to be published on Linear Partial Differential Equations with a Continuous Infinitude of Variables.

[5] This formula has already been obtained by Volterra as the general solution of a functional differential equation in his paper, *Atte R. Accad. Lincei*, Ser. 6, **23**, 1914, 1st semester, (p. 393.)

[6] See paper by writer entitled, Integro-differential Equations with Constant Kernels, *Bull. Amer. Math. Soc.*, **26**, 1920 (193).

THERMOKINETICS OF LIOMETOPUM APICULATUM MAYR

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Variation in the activity of a cold-blooded animal is largely dependent on metabolic changes, which in turn probably depend mainly on the accelleration of oxidation and of other chemical reactions. The physical nature of the environment affects most of these chemical processes, and we should expect that the same physical properties would also directly influence the kinetic manifestations of animal life. While many observations record the qualitative relation of animal activity to such factors as humidity and external temperature, only a few give definitive numerical results.

An opportunity for precise quantitative measurement of the variation in the rate of expenditure of animal energy under the different natural stimuli arising from changing meteorological conditions, is afforded by the study of the foraging and patrolling activities of ants of the subfamily Dolichoderinae. Although some of my observations have been made on other genera of this group (*Tapinoma*, *Dorymyrmex*), I find the most suitable material to be the "Argentine ant," *Iridomyrmex humilis* Mayr, and two species of *Liometopum*. In briefly describing the activities of these ants and the observations based upon them, a number of reasons are proposed in the following paragraphs why members of the subfamily, and particularly the California *Liometopa*, are thought to be better adapted than nearly any other organism for the close quantitative investigation of kinetic response to changes in field conditions.

The trail-running habit, which is common to all genera of Dolichoderinae except *Leptomyrmex* of Australia,[1] is fundamental for the observations discussed below. It permits permanent observing stations to be set up along the trail so that the speed over fixed intervals of distance can be